Nonlinear Dynamical Response of Impulsively Loaded Structures: A Reduced Basis Approach

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This paper deals with predicting the nonlinear dynamic response of a structure under impulsive loading by means of reduction methods. Techniques for reducing the number of equations to be solved, e.g., when such equations come from a finite element model, are referred to as reduction methods. The order of a dynamical system is reduced by a Rayleigh-Ritz technique using selective basis vectors. Numerical results are obtained using Ritz vectors and their derivatives as basis vectors. Both use of such derivatives and use of updating basis vectors provide excellent results. The proposed reduction technique is enhanced by updating the stiffness matrix with a reasonable reassembly frequency.

Nomenclature

C = damping matrix for the structure
$\bar{C}$ = reduced damping matrix for the structure
ϵ = error norm, defined by Eq. (20)
I_t = total unbalanced force vector at time t
K = stiffness matrix for the structure
$\bar{K}$ = reduced stiffness matrix for the structure
$K^{(d)}$ = stiffness matrix with change in sth term of Z
K^* = effective stiffness matrix
$\bar{K}^*$ = reduced effective stiffness matrix
M = mass matrix for the structure
$\bar{M}$ = reduced mass matrix for the structure
m = number of basis vectors
N_f = frequency of updating stiffness
N_u = frequency of updating basis vectors
n = total number of degree-of-freedom in the finite element model
P = applied loads vector
P^* = effective loads vector
$\bar{P}^*$ = reduced effective loads vector
t_d = time duration of impulsive load
Δt = time integration step
U = displacement vector
$\dot{U}$ = velocity vector
$\ddot{U}$ = acceleration vector
$U^{(d)}$ = displacement vector with small change in sth term of Z
V_c = basis vector related to current state
V_i = basis vector related to initial state
Z = reduced displacement vector
$Z^{(d)}$ = reduced displacement vector with small change in sth term
z_s = sth components of reduced displacement vector
Ψ = matrix of basis vectors
Ψ_i = ith basis vector
τ_i = ith natural period of structure

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Introduction

ADVANCES in nonlinear dynamical analysis have been in 1) finding a general formulation to modify the tangent stiffness matrix due to geometric changes (large displacements or large strains) or material nonlinearities, 2) choosing proper time integration schemes to predict accurately the dynamical response, and 3) using a better iteration procedure to accelerate convergence. The computer cost for performing the nonlinear dynamical response of a large structural system is especially significant. Techniques for reducing the degrees of freedom of a large dynamical system have been proposed to solve the nonlinear dynamical problem.

The nonlinear dynamical response of a structure is usually solved either by the direct time integration method or by the modal superposition method. Mathematically, the latter method involves a coordinate transformation from the finite element displacements to the reduced (or generalized) displacements. Traditionally, eigenvectors of a dynamical system are used as basis vectors (or modes) of the complete system. In this approach, a set of n coupled equations of motion is transformed into a set of $m(m \ll n)$ uncoupled equations. Each uncoupled equation can be solved by any suitable integration method. The nodal displacements and stresses are then given as the superposition of the normal modes. The modal superposition method for solving nonlinear dynamical response requires extensive computational effort to modify the eigenvectors as the solution progresses. However, this method has the advantage that a greatly reduced set of equations is solved. If a few modes need to be considered during the response and the selected modes can be updated by an efficient algorithm, the method can compete with the direct time integration method.

Wilson et al.[1] suggest that Ritz vectors, which are not the exact modes of a given dynamical system, are a more efficient means of transforming a dynamical system into its reduced (or generalized) coordinates. Their work applied to the transient response of a linear system; in some cases, more accurate results were predicted with Ritz vectors as basis vectors than with exact eigenvectors as basis vectors. However, the generation of Ritz vectors avoids the lengthy computation needed to generate exact eigenvectors. They solved for the response of typical steel frame structures with Ritz vectors. Competitive numerical results were obtained with a subspace of Ritz vectors rather than a subspace of exact eigenvectors.

McNamara and Marcal[2] applied the corrected incremental equations of motion; i.e., the corrected (or unbalanced) forces related to the previous step were added to the incremental applied forces vector, to obtain solutions. They concluded that the corrected incremental scheme gave more stable and accurate solutions than the simple incremental equations of motion. McNamara[3] used the incremental load-corrected equation to predict nonlinear dynamical response of structures modeled by classical beam elements. He showed that accurate and economic solutions can be obtained by only periodically updating the stiffness. Mondkar and Powell[4] investigated the static and dynamic response of nonlinear structures by both step-by-step and iteration schemes. The nonlinear solutions of cantilever beam, clamped beam, and shallow arch modeled by isoparametric elements were solved by various schemes. References 2-4 used various direct time integration schemes to solve for the nonlinear dynamical response of structures.

Horri and Kawahara[5] used the modal superposition method to solve nonlinear dynamical problems of simple structures subjected to harmonic excitation. Eigenvalues and the associated eigenvectors were updated at every time increment. Nickell[6] employed a local modal superposition principle to solve nonlinear dynamical problems. The principle states that small harmonic motion may be superimposed on large static motion, and that small forced motion may be represented in terms of the nonlinear (tangent stiffness) frequency spectrum. The subspace iteration scheme was employed to update eigenvectors at each time step. Geometrically nonlinear problems were solved by the principle of local modal superposition. Bathe and Gracewski[7] applied modal superposition and substructuring techniques to solve structural vibration problems. Structures with slightly nonlinear stiffness and local nonlinearity were used as examples to demonstrate the technique. Noor[8] applied the lowest vibration modes of the initial state of the structure and of the nonlinear steady state of the structure as basis vectors to predict the nonlinear dynamical response of step loading on a clamped, shallow, spherical cap subjected to a point load at the apex. It was shown that reasonably accurate solutions can be obtained by the reduction method. Idelsohn and Cardona[9] augmented Ritz vectors by adding derivatives of the vectors with respect to generalized displacements to treat nonlinear dynamical problems. An error measure that indicates the need of performing a basis updating is proposed. References 5-9 were limited to treating nonlinear dynamical response of either step or harmonic loads by various reduction techniques.

The present research deals with predicting the nonlinear dynamical response of a structure under impulsive loading by means of reduction methods. A set of orthogonal Ritz vectors[1] and their derivatives, which are not the exact eigenmodes of a given dynamical system, are chosen as basis vectors. It has not been proven that the modal superposition method using exact eigenmodes produces better results than those with other well-conceived sets of orthogonal vectors. Numerical algorithms are presented for generating and updating the Ritz vectors and their derivatives. These algorithms have been developed to provide for basis vector updating that may be required to obtain the response to impulsive loads. Ritz vectors have two primary advantages over eigenvectors: 1) They are easier to generate because of fewer computational operations than needed for eigenvectors. The reduction of computer cost is greater for a large structural system. 2) They can account for the spatial load distribution, which the generation of eigenvectors ignores. To account for nonlinearities of the structural system, derivatives of basis vectors with respect to the generalized coordinates are added to the basis vectors. Additionally, the basis set can be updated on the basis of an error criteria.

The object of the research is, therefore, to generate basis vectors in a computationally efficient manner so that nonlinear dynamical response, due to impulsive loads, can be predicted with accuracy. To enable comparisons to previous work, only geometric nonlinearity is considered.

Nonlinear Formulation

The equation of motion governing the nonlinear dynamical system can be written in incremental form as

$$M\Delta\ddot{U}_{t+\Delta t} + C_t\Delta\dot{U}_{t+\Delta t} + K_t\Delta U_{t+\Delta t} = \Delta P_{t+\Delta t} + I_t \tag{1}$$

and

$$I_t = P_t - M\ddot{U}_t - C_t\dot{U}_t - F_t$$

where $M =$ the mass matrix, C_t the damping matrix at time t, $K_t =$ the nonlinear stiffness matrix at time t, $\Delta P_{t+\Delta t} =$ the incremental applied loads vector at time $t + \Delta t$, $I_t =$ the total unbalance force vector at time t, $F_t =$ vector of nodal point forces corresponding to the internal element stress at time t, and $\Delta\ddot{U}_{t+\Delta t}$, $\Delta\dot{U}_{t+\Delta t}$ and $\Delta U_{t+\Delta t}$ the unknown incremental acceleration, velocity, and displacement vectors at time $t + \Delta t$. The constant-average-acceleration method (also called the trapezoidal rule) of the Newmark integration scheme is considered here because of its relatively good stability and accuracy characteristics. The basic assumptions of the trapezoidal rule are

$$\dot{U}_{t+\Delta t} = \dot{U}_t + \frac{\Delta t}{2}\left(\ddot{U}_{t+\Delta t} + \ddot{U}_t\right) \tag{2a}$$

$$U_{t+\Delta t} = U_t + \frac{\Delta t}{2}\left(\dot{U}_{t+\Delta t} + \dot{U}_t\right) \tag{2b}$$

using Eq. (2) and the relation $\Delta U_{t+\Delta t} = U_{t+\Delta t} - U_t$, Eq. (1) gives

$$K^*_{t+\Delta t}\Delta U_{t+\Delta t} = P^*_{t+\Delta t} \tag{3}$$

Here, the effective stiffness matrix $K^*_{t+\Delta t}$ is given by

$$K^*_{t+\Delta t} = K_t + \frac{2C_t}{\Delta t} + \frac{4M}{\Delta t^2} \tag{4}$$

and the effective load vector $P^*_{t+\Delta t}$ is defined as

$$P^*_{t+\Delta t} = \Delta P_{t+\Delta t} + I_t + M\left(\frac{4\dot{U}_t}{\Delta t} + 2\ddot{U}_t\right) + 2C_t\dot{U}_t \tag{5}$$

Suppose the displacement increment ΔU can be approximated by m linearly independent basis vectors, giving the coordinate transformation

$$\Delta U = \Psi Z(t) \tag{6}$$

where Ψ is a matrix composed of $m (m \ll n)$ basis vectors and $Z(t)$ represents a set of reduced coordinates. Using the Rayleigh-Ritz technique, we can obtain the governing equations of motion for the reduced system

$$\bar{K}^*_{t+\Delta t}Z = \bar{P}^*_{t+\Delta t} \tag{7}$$

Here, the reduced effective matrix $\bar{K}^*_{t+\Delta t}$ is given by

$$\bar{K}^*_{t+\Delta t} = \Psi^T K^*_{t+\Delta t}\Psi \tag{8}$$

or

$$\bar{K}^*_{t+\Delta t} = \frac{4}{\Delta t^2}\bar{M} + \frac{2}{\Delta t}\bar{C}_t + \bar{K}_t \tag{9}$$

where

$$\bar{K}_t = \Psi^T K_t \Psi \tag{10a}$$

$$\bar{C}_t = \Psi^T C_t \Psi \tag{10b}$$

$$\bar{M} = \Psi^T M \Psi \tag{10c}$$

The reduced effective load vector $\bar{P}^*_{t+\Delta t}$ is written as

$$\bar{P}^*_{t+\Delta t} = \Psi^T P^*_{t+\Delta t} \tag{11}$$

Equation (7) can be solved by virtually any suitable numerical method; remember that this is a very small equation set because the efficiency comes from using a small number of basis vectors. The approximate incremental displacement response for the complete model is then obtained by the transformation in Eq. (6).

Some of the possible candidates for basis vectors would include eigenvectors, Ritz vectors, and derivatives of the above vectors with respect to the generalized coordinates. Numerical evaluation of the derivatives of the basis vectors requires definition of the change in the reduced displacement vector due to a change in the sth term

$$Z^{(d)} = (0,0, \ldots, \Delta z_s, \ldots, 0) \tag{12}$$

where the components of $Z^{(d)}$ equal zero, except in the sth term. Note that Δz_s should be chosen small enough to accurately estimate the derivatives

$$\Delta U^{(d)} = \Psi Z^{(d)} \tag{13}$$

where $\Delta U^{(d)}$ is the incremental displacement vector due to a small change in sth term of the reduced displacement vector. This gives a total displacement vector reflecting the small change in the sth term as

$$U^{(d)} = U_t + \Delta U^{(d)} \tag{14}$$

Now the stiffness matrix, based on total displacement vector $U^{(d)}$, is obtained through a routine finite element program. We have

$$\Delta K = \mathrm{K}^{(d)} - K_t \tag{15}$$

here, ΔK is the change in the 'global' stiffness due to a change in the reduced displacement term Δz_s. Now define the approximate derivative of the basis vector with the use of a finite-difference representation. Then the simplified formulation can be written as

$$\frac{\partial \Psi_r}{\partial z_s} \approx -K_t^{-1} \frac{\Delta K}{\Delta z_s} \Psi_r \tag{16}$$

and K_t^{-1} is obtained by an LDL^T decomposition of the current stiffness matrix. As suggested in Ref. 9. the inertial terms are ignored in determining the derivatives.

Updating Basis

For the initial state ($t = 0$), the basis vectors are generated as

$$K_t \Psi_r^* = P \qquad r = 1 \tag{17}$$

and

$$K_t \Psi_r^* = M \Psi_{r-1} \qquad r > 1 \tag{18}$$

The vectors are orthonormalized by the following procedure

$$\Psi_r' = \Psi_r^* - \sum_{i=1}^{r-1} c_i \Psi_i \tag{19}$$

where

$$c_i = \Psi_i^T M \Psi_r^* \tag{20}$$

$$\Psi_r = \frac{\Psi_r'}{\Psi_r'^T M \Psi_r'} \tag{21}$$

A weighted error norm of the unbalance force vector[9] showing the need for updating basis vectors is used as an error tolerance in the following examples. The error norm is represented as

$$e = \frac{1}{n} \frac{||I_{t+\Delta t}||}{(||P_{t+\Delta t}|| + ||M\ddot{U}_{t+\Delta t}||)} \tag{22}$$

where n represents the number degrees of freedom of the full system; $||I_{t+\Delta t}||$ is the norm of the unbalance force vector; $||P_{t+\Delta t}||$ is the norm of the total applied load vector at time $t + \Delta t$; and $||M\ddot{U}_{t+\Delta t}||$ is the norm of the inertial force at time $t + \Delta t$. As the solution proceeds in time, the numerator in Eq. (22) increases because the governing Eq. (7) depends upon the basis vectors, and, of course, the basis vectors depend upon the current state. If the basis vectors do not well-represent the displacements vector, the misrepresentation leads to errors in the unbalance force vector. This is because the unbalance forces are calculated using the local (element) stiffness, which depends upon the current total displacement state. The denominator in Eq. (22) serves to normalize the error norm, i.e., to compare the unbalance to some definition of total loading. Note that including the inertial term in Eq. (22) gives the error norm utility in the residual response regime.

If the error norm e is less than a prescribed tolerance, the basis vectors are not updated; otherwise, some new basis vectors related to the current state can be added to the basis vectors. For forced-vibration era ($t \le t_d$), the generation of updated Ritz vectors is the same as the generation of initial set of Ritz vectors. For residual vibration era, the first basis vector of the current state is generated from a modification of Eq. (17) as

$$K_t \Psi_r^* = M \Psi_{1,0} \tag{23}$$

where $\Psi_{1,0}$ is the first basis vector of the initial state. The generation of additional basis vectors proceeds in the same manner as used in generating the initial state, i.e., Eq. (18) to Eq. (21). When updating the basis vectors, only the basis vectors related to the current state are updated; the basis vectors related to the initial state remain unchanged.

Some consideration has been given to methods of ranking the basis vectors. Since the Ritz vectors reflect the spatial load distribution, it would seem that the effect of ranking would not be as significant as in use of the modal method. A possible approach to ranking the Ritz vectors is obtained by definition of a ranking parameter λ_i. During forced response ($t \le t_d$)

$$\lambda_i = \frac{\Psi_i^T P}{\Psi_i^T K_t \Psi_i} \tag{24}$$

Here, $\Psi_i^t P$ is similar to a modal force vector, $\Psi_i^t K_t \Psi_i$ is like a modal stiffness. Therefore, ranking parameter λ_i gives a measure of the relative response in each vector basis space. In that, the ranking parameter given in Eq. (24) is similar to a modal participation factor, except that here Ritz vectors rather than eigenvectors are needed to calculate λ_i. During residual response ($t > t_d$)

$$\lambda_i = \Psi_i^T M U_t \tag{25}$$

where $\Psi_i^T M U_t$ is much like a modal initial condition at time t. In Eq. (25), the intent is to base ranking on the effect of the inertia loading.

Numerical Examples

Clamped Beam

Results for the dynamical response of a clamped beam subjected to an impulsive load are presented in Figs. 1 to 8. The finite element model of the beam consists of five 8-node plane stress elements in the half span. The clamped beam under step loading was analyzed by Mondkar et al.[4] by using various direct integration techniques for an unreduced set of equations. The values of the parameters used for the beam are

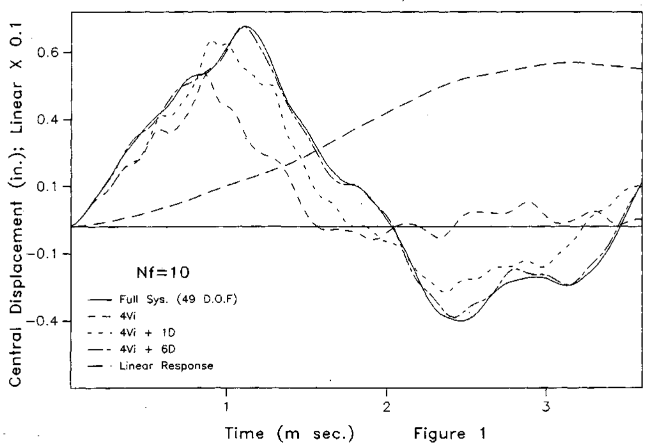

Fig. 1 Displacement of Clamped Beam – Basis Set 1.

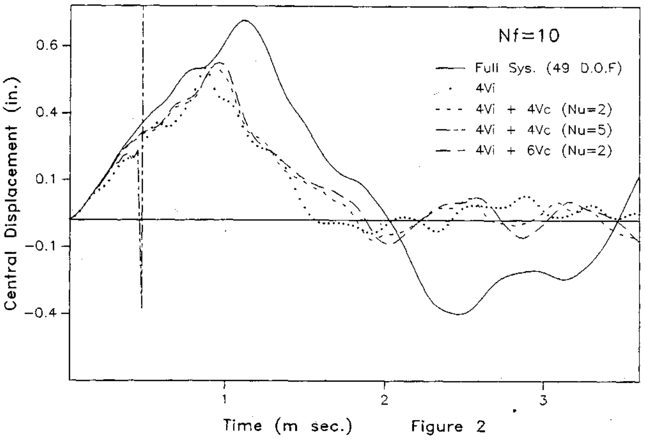

Fig. 2 Displacement of Clamped Beam – Basis Set 2.

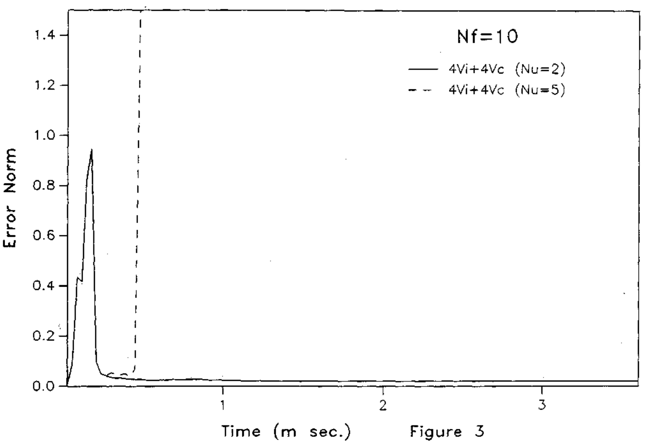

Fig. 3 Error Norm for Clamped Beam – Basis Set 2.

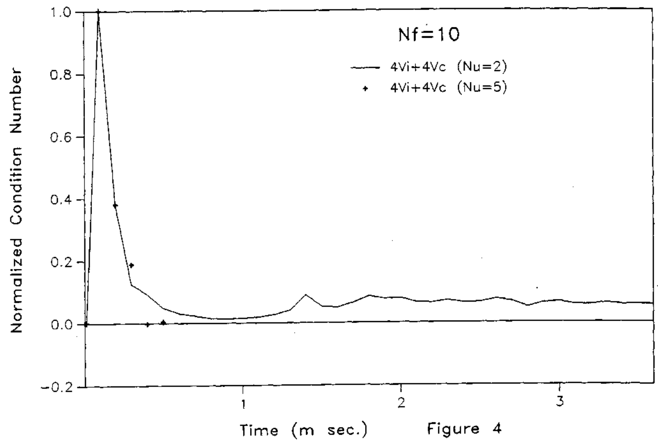

Fig. 4 Condition Number for Clamped Beam – Basis Set 2.

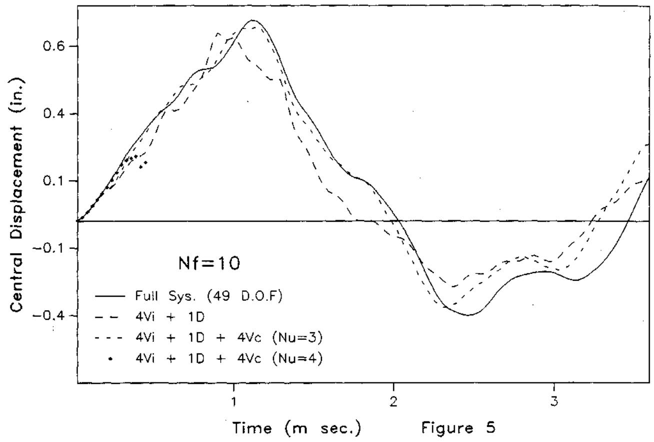

Fig. 5 Displacement of Clamped Beam – Basis Set 3.

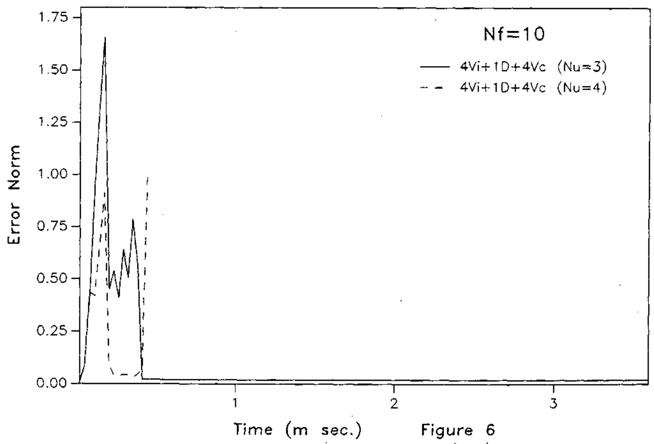

Fig. 6 Error Norm for Clamped Beam – Basis Set 3.

taken to be: total length = 20 in.; cross-sectional area = 1/8 in. × 1 in.; Young's modulus = 30×10^6 psi; Poisson's ratio = 0.0; density = 0.098 lb/in.^{3}; centrally applied load = 640 lbs; time duration of load = 1.8 ms which is about 20% of the fundamental natural period of the undeformed beam; time integration step = 10 μs. For these solutions, the stiffness matrix is only updated every tenth time increment. In Fig. 1, the first basis set used four 'initial state' Ritz vectors, the second basis set used four 'initial state' Ritz vectors and one derivative vector, the third set used four 'initial state' Ritz vectors and six derivative vectors, i.e., derivatives relating to the first three Ritz vectors. Results in Fig. 1 clearly demonstrate the benefit of including more derivatives in the basis set. In Fig. 2 'current state' Ritz vectors are added to the initial basis set, which includes only 'initial state' Ritz vectors. In this case, the 'initial state' included four vectors. The second set is augmented by four current Ritz vectors that are updated by a loose error criteria. The third set is augmented by the same number of current Ritz vectors, but is updated by tightening the error criteria to the extent that the frequency of updating the Ritz vectors becomes equal to the frequency of updating the stiffness matrix. Finally, the fourth basis vector set is augmented by six current Ritz vectors updated by a loose error criteria. Note that when a violation of the error norm occurs, the algorithm is programmed not to update the basis set until the stiffness matrix is updated. Here, the second set required two updatings and the fourth set required two updatings as well. These results indicate that simply adding more current

Ritz vectors does not significantly improve the results. Furthermore, when the frequency of updating the basis set approaches the frequency of stiffness matrix updating, the response becomes unstable. The error norm vs time response for the second and third basis set is shown in Fig. 3. Also, a normalized condition number (the ratio between the maximum and minimum eigenvalues of the reduced systems) vs time is shown in Fig. 4. The condition number can serve to determine whether or not numerical precision is acceptable. In these solutions, the precision does not present a problem. However, in the case of the unstable response, the lowest eigenvalue of the reduced system becomes negative, resulting in a negative definite reduced equation set.

The response curves presented in Fig. 5 are similar to those in Fig. 2, except one derivative vector is added to the basis. The strategy used to update current Ritz vectors is identical to that used to obtain the results in Fig. 2. The error norm and condition number are given in Figs. 6 and 7. In comparing the response curves in Fig. 5 to those in Fig. 2, it is noted that the addition of one derivative vector to the basis set results in the basis updating being much more beneficial. In fact, one derivative with four current vectors added to the initial basis set, gives similar results to those obtained by adding six derivatives to the initial basis set. Again, when the basis is updated at the same frequency as the stiffness matrix, the solution becomes unstable. In this case, the condition number becomes negative.

A different model of the clamped beam, i.e., using ten elements in the whole span, provides the results given in Fig. 8. The response curves given in Fig. 8 are based on three different basis sets. The first set included Ritz vectors and their derivatives without ranking, the second includes eigenvectors and their derivatives without ranking, and, finally, the third set is similar to the second but with ranking. The derivatives are taken after ranking the basis vectors. The results obtained using Ritz vectors without ranking are close to those using eigenvectors with ranking. For this particular loading, the displacement response is symmetric with respect to the center of the beam. Only symmetric modes contribute to the response. When checking Ritz vector shapes, it is observed that the first four Ritz vectors are similar to the first four symmetric eigenvectors. It is interesting in this case to note that the algorithm used to generate the first few Ritz vectors has the effect of ranking the basis vectors.

In this case, therefore, the Ritz vectors do not need to be rank ordered. For problems requiring that higher frequency content be represented, a larger number of basis vectors would be needed to obtain accurate solutions. Furthermore, the ranking previously discussed could then easily reorder the basis vector set and prove useful to providing efficient solutions.

Shallow Arch

Figures 9 to 14 show the dynamic response of a shallow arch subjected to an impulsive load. The finite element model of the arch consists of eight 8-node plane stress elements in the half span. The same problem was studied by Mondkar and Powell,[4] in which static response was predicted. The values of the parameters used in shallow arch are as follows: Young's modulus $= 10 \times 10^6$ psi; Poisson's ratio $= 0.2$; half-subtended angle of the arch $= 7.3373$ deg; radius of curvature of the arch $= 133.1138$ in.; thickness $= 0.188$ in.; density $= 0.09467$ lb/in.[3]; concentrated load at center $= 30$ lb; time duration of impulsive load $= 2.4$ ms, which is about 40% of the fundamental natural period of the undeformed arch; time integration step $= 30$ μs. The present model has been verified by compar-

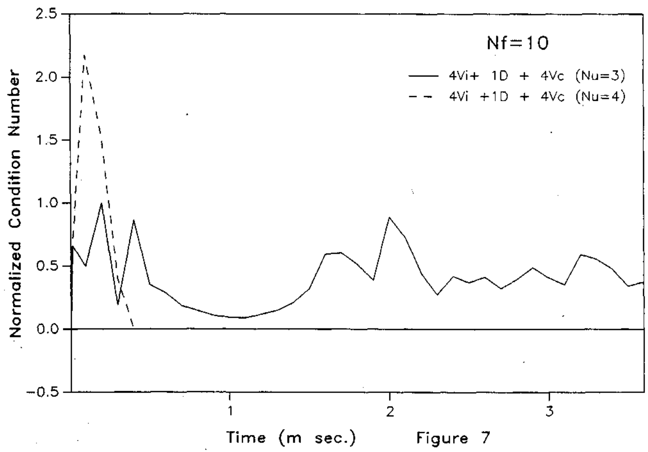

Fig. 7 Condition Number for Clamped Beam – Basis Set 3.

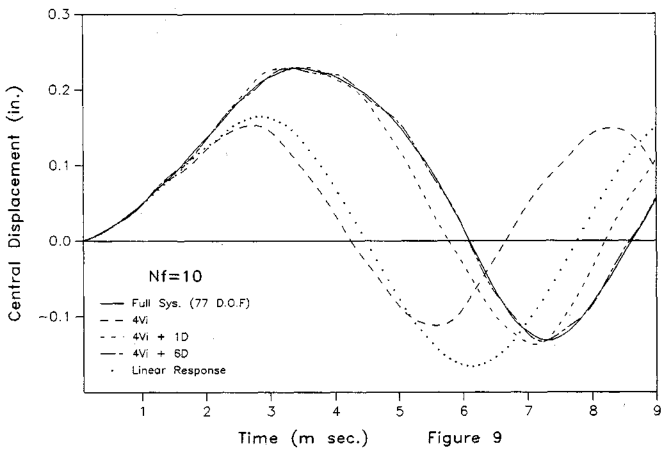

Fig. 9 Displacement of Shallow Arch – Basis Set 1.

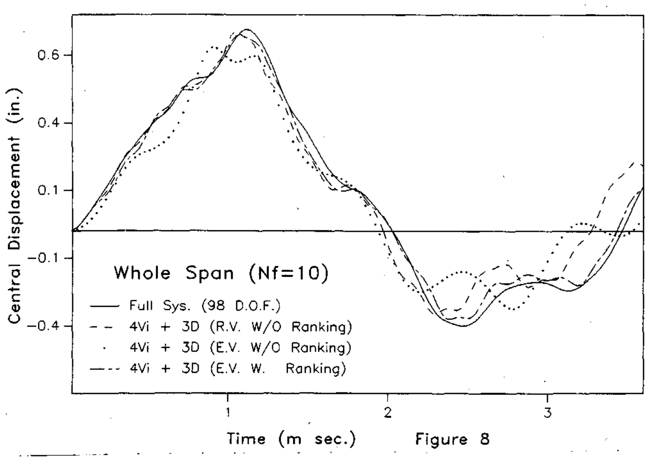

Fig. 8 Displacement of Clamped Beam – Different Basis Sets.

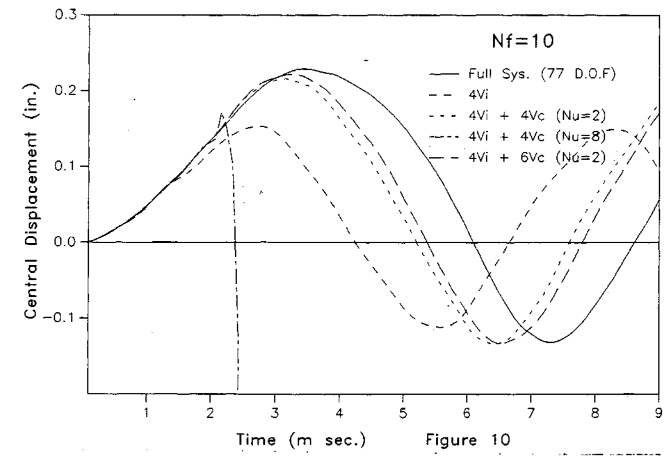

Fig. 10 Displacement of Shallow Arch – Basis Set 2.

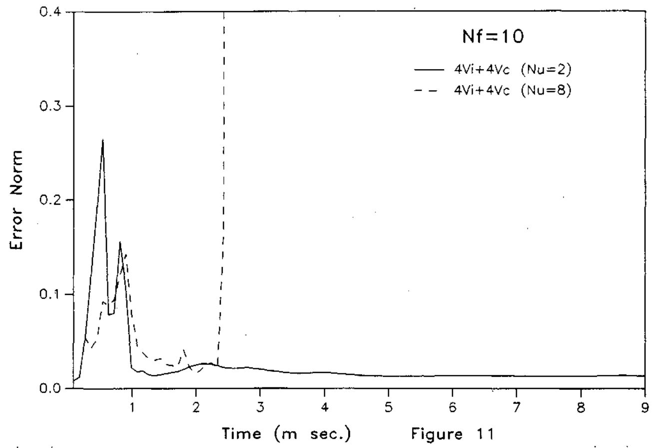

Fig. 11 Error Norm for Shallow Arch – Basis Set 2.

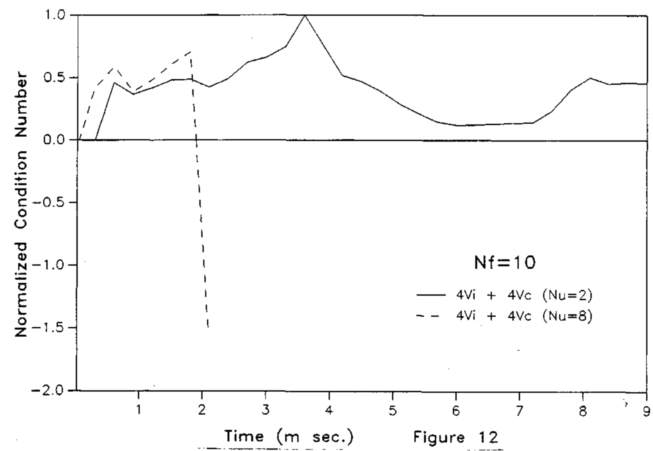

Fig. 12 Condition Number for Shallow Arch – Basis Set 2.

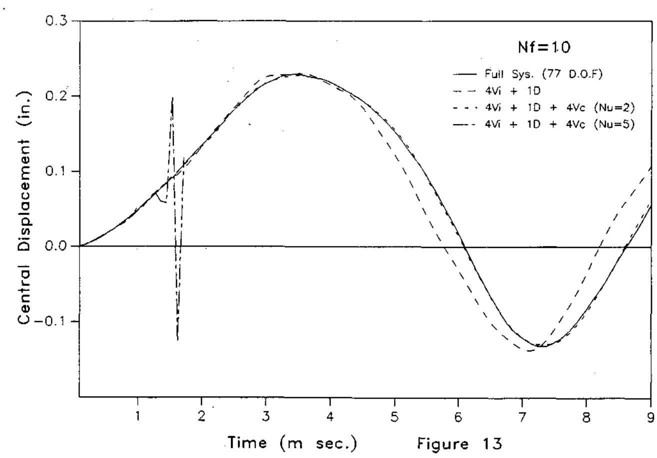

Fig. 13 Displacement of Shallow Arch – Basis Set 3.

ing the static response prediction to that obtained in Ref. 4. The combination of basis vectors used in this example are identical to those selected for the clamped beam. For the nonlinear response shown in Fig. 9, results have been obtained for basis sets with a different number of derivatives. As in the previous example, the advantage of using the derivatives is evident. A comparison of response curves solved by adding more current Ritz vectors or by tightening the error criteria is shown in Fig. 10. Again, the results show that simply adding more current Ritz vectors does not significantly improve the solution. Also, the frequency of updating the basis set can have an adverse effect. The response curve of error norm vs time, based on the second and third basis sets in Fig. 10, is

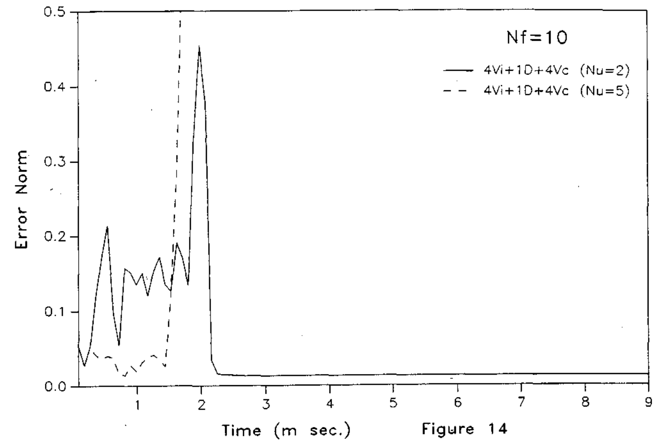

Fig. 14 Error Norm for Shallow Arch – Basis Set 3.

presented in Fig. 11. The normalized condition number vs time response is shown in Fig. 12. Again, the unstable response curve shown in Fig. 10 is related to an ill-conditioning, i.e., negative condition number, of the reduced equation set. In Fig. 13, again the advantages of updating the basis in combination with limited derivative information is evident. Selected error norms are given in Fig. 14.

Conclusions

In this work, initial Ritz vectors, derivative of Ritz vectors, and updated Ritz vectors have been chosen as candidates for basis vectors. Results indicate that the response predicted using basis vectors with a suitable number of derivatives closely follows results obtained for the full system. Updating the basis vectors can also provide accurate solutions, provided that derivative information is utilized to a limited extent. Efficiency is enhanced by updating the stiffness matrix at some predetermined reassembly frequency. The results would indicate any unstable condition can be avoided by adequately separating the basis vector update vs stiffness matrix update frequencies. It may be possible to use the condition number to avoid any unstable response behavior. Due to the computational efficiency of the reduced basis approach presented here, it is suggested that this approach is a viable alternative to either direct or modal solutions.

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